Discrete Mathematics 312 (2012) 2140–2145

Contents lists available at SciVerse ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Improved degree conditions for Hamiltonian properties

Arnfried Kemnitz [a,*], Ingo Schiermeyer [b]

[a] *Computational Mathematics, Technische Universität Braunschweig, 38023 Braunschweig, Germany*

[b] *Institut für Diskrete Mathematik und Algebra, Technische Universität Bergakademie Freiberg, 09596 Freiberg, Germany*

ARTICLE INFO

Article history:
Available online 10 August 2011

Keywords:
Hamiltonian
Hamiltonian-connected
Traceable
Bondy's Theorem
Degree sum conditions

ABSTRACT

In 1980, Bondy proved that for an integer $k \geq 2$ a $(k+s)$-connected graph of order $n \geq 3$ is traceable ($s = -1$) or Hamiltonian ($s = 0$) or Hamiltonian-connected ($s = 1$) if the degree sum of every set of $k+1$ pairwise nonadjacent vertices is at least $\frac{1}{2}((k+1)(n+s-1)+1)$. This generalizes the well-known sufficient conditions of Dirac ($k = 0$) and Ore ($k = 1$). The condition in Bondy's Theorem is not tight for $k \geq 2$. We improve this sufficient degree condition and show the general tightness of this result.

1. Introduction

Let G be a finite and simple graph with vertex set $V(G)$ and edge set $E(G)$. A path or cycle of a graph is called *Hamiltonian* if it contains each vertex of the graph exactly once. A graph is called *traceable* if it contains a Hamiltonian path, *Hamiltonian* if it contains a Hamiltonian cycle, and *Hamiltonian-connected* if it contains a Hamiltonian path between each pair of vertices. The *r-closure* $\mathrm{Cl}_r(G)$ is obtained from G by recursively joining pairs of nonadjacent vertices with degree sum at least r as long as such pairs do exist.

In 1952, Dirac proved a sufficient condition for Hamiltonicity, which was extended by Ore.

Theorem 1 (*Dirac 1952 [5], Ore 1960 [7]*). *If G is a graph of order $n \geq 3$ and minimum degree $\delta(G)$ with $\delta(G) \geq (n+s)/2$, then G is traceable for $s = -1$, Hamiltonian for $s = 0$, and Hamiltonian-connected for $s = 1$.*

The following well-known degree sum condition by Ore generalizes the degree condition of Theorem 1.

Theorem 2 (*Ore 1960/63 [7,8]*). *Let G be a graph of order $n \geq 3$. If the degree sum $d(u) + d(v) \geq n + s$ for all pairs u, v of nonadjacent vertices, then G is traceable for $s = -1$, Hamiltonian for $s = 0$, and Hamiltonian-connected for $s = 1$.*

This was improved by Chvátal and Erdős, who proved the following result.

Theorem 3 (*Chvátal, Erdős 1972 [4]*). *If G is a graph of order $n \geq 3$ with independence number $\alpha(G)$ and connectivity $\kappa(G)$ with $\alpha(G) \leq \kappa(G) - s$, then G is traceable for $s = -1$, Hamiltonian for $s = 0$, and Hamiltonian-connected for $s = 1$.*

Note that the results of Theorems 1–3 are tight in the sense that the conditions of the theorems cannot be weakened.

Theorem 1: The graph $G \cong K_{\lfloor \frac{n+s-1}{2} \rfloor} + \lceil \frac{n-s+1}{2} \rceil K_1$ satisfies $\delta(G) = \lfloor \frac{n+s-1}{2} \rfloor$ but G is non-traceable for $s = -1$, non-Hamiltonian for $s = 0$, and non-Hamiltonian-connected for $s = 1$.

* Corresponding author. Tel.: +49 0 531 3917516; fax: +49 0 531 3917414.
E-mail addresses: a.kemnitz@tu-bs.de (A. Kemnitz), Ingo.Schiermeyer@tu-freiberg.de (I. Schiermeyer).

0012-365X/$ – see front matter
doi:10.1016/j.disc.2011.07.013

Theorem 2: The graph $G \cong (K_{n-s-2} \cup K_1) + K_{s+1}$ satisfies $d(u) + d(v) \le (n-2) + (s+1) = n+s-1$ for all pairs u, v of nonadjacent vertices but G does not have the corresponding property.

Theorem 3: The graph $G \cong K_{r,r+s-1}$ satisfies $\alpha(G) = \kappa(G) - s + 1$ but G does not have the corresponding property.

Bondy and Chvátal proved the following theorem.

Theorem 4 (*Bondy, Chvátal 1976 [3]*)**.** *Let G be a graph of order n. The graph G is traceable, Hamiltonian, or Hamiltonian-connected if and only if $\mathrm{Cl}_{n-1}(G)$ is traceable, $\mathrm{Cl}_n(G)$ is Hamiltonian, or $\mathrm{Cl}_{n+1}(G)$ is Hamiltonian-connected, respectively.*

In the conditions of Theorems 1 and 2, independent sets of vertices of order 1 and 2 are considered. Bondy [2] proved sufficient conditions for the Hamiltonicity of graphs with respect to independent sets of arbitrary size.

Let $\sigma_k(G)$ be the minimum sum of the degrees of any k independent vertices, that is,

$$\sigma_k(G) = \min\left\{\sum_{v \in S} d(v) : S \subseteq V(G), S \text{ independent}, |S| = k\right\}.$$

If G does not contain k independent vertices, then we set $\sigma_k(G) = \infty$.

Theorem 5 (*Bondy 1980 [2]*)**.** *For $k \ge 2$ let G be a $(k+s)$-connected graph of order $n \ge 3$. If*

$$\sigma_{k+1}(G) \ge \frac{1}{2}((k+1)(n+s-1)+1)$$

then G is traceable for $s = -1$, Hamiltonian for $s = 0$, and Hamiltonian-connected for $s = 1$.

Setting $k = 0$ in Theorem 5 results in Dirac's Theorem (Theorem 1) and $k = 1$ in Ore's Theorem (Theorem 2).

It was asked in [1] for general tightness results of Theorem 5.

We show that Theorem 5 is not tight for $n + s$ even by improving the degree condition of the theorem.

2. Main theorem

Theorem 6. *Let G be a $(k+s)$-connected graph of order $n \ge 3$ and let $2 \le k \le \frac{n-s-2}{2}$. If*

$$\sigma_{k+1}(G) \ge \frac{1}{2}((k+1)(n+s-1)+1) \quad \text{for } n+s \text{ odd}$$

and

$$\sigma_{k+1}(G) \ge \begin{cases} \frac{1}{2}((k+1)(n+s-1)+3-k), & k = \dfrac{n-s-2}{2} \\ \frac{1}{2}((k+1)(n+s-1)+1-k), & k < \dfrac{n-s-2}{2} \end{cases} \quad \text{for } n+s \text{ even}$$

then G is traceable for $s = -1$, Hamiltonian for $s = 0$, and Hamiltonian-connected for $s = 1$.

The degree condition of Theorem 6 is tight.

For odd $n+s$, the graph $G \cong K_{\frac{n+s-1}{2}} + \frac{n-s+1}{2}K_1$ fulfills $\sigma_{k+1}(G) = (k+1)\left\lfloor\frac{n+s-1}{2}\right\rfloor$ for $2 \le k \le \frac{n-s-1}{2}$ but is non-traceable for $s = -1$, non-Hamiltonian for $s = 0$, and non-Hamiltonian-connected for $s = 1$.

For even $n+s$, the graph $G \cong \left(\frac{n-s-2}{2}K_1 \cup K_2\right) + K_{\frac{n+s-2}{2}}$ fulfills $\sigma_{k+1}(G) = (k+1)\left\lfloor\frac{n+s-1}{2}\right\rfloor + \left\lfloor\frac{k}{\frac{n-s-2}{2}}\right\rfloor$ for $2 \le k \le \frac{n-s-2}{2}$ but does not have the corresponding Hamiltonian property.

Note that for $n + s$ odd, the conditions of Theorems 5 and 6 coincide. Nevertheless, we will present a short proof of this case. If $n + s$ is even, then the difference between the right-hand sides of the conditions equals $\frac{k}{2}$.

3. Proof of the main theorem

1. *$n + s$ odd*

Assume that G contains a set I of $k+1$ pairwise independent vertices $v_0, \ldots, v_k$ with $d(v_i) + d(v_j) \le n+s-1$ for all $0 \le i < j \le k$. Then

$$k \cdot \sum_{0 \le i \le k} d(v_i) = \sum_{0 \le i < j \le k} (d(v_i) + d(v_j)) \le \binom{k+1}{2}(n+s-1) = \frac{1}{2}k(k+1)(n+s-1)$$

which implies $\sigma_{k+1}(G) \le \sum_{i=0}^{k} d(v_i) \le \frac{1}{2}(k+1)(n+s-1)$, a contradiction to the condition of Theorem 6.

It follows that there exists a pair of vertices having degree sum at least $n+s$ in each set I of $k+1$ independent vertices, that is, the closure $\mathrm{Cl}_{n+s}(G)$ does not contain an independent set of $k+1$ vertices. Therefore, $\alpha(\mathrm{Cl}_{n+s}(G)) \le k \le \kappa(\mathrm{Cl}_{n+s}(G)) - s$

since $\mathrm{Cl}_{n+s}(G)$ is $(k+s)$-connected. Then, by Theorem 3, the graph $\mathrm{Cl}_{n+s}(G)$ is traceable for $s=-1$, Hamiltonian for $s=0$, and Hamiltonian-connected for $s=1$. Application of Theorem 4 completes the proof. □

2. *$n+s$ even*

2.a *Hamiltonicity* ($s=0$)

Let C be a cycle of maximum length in a graph G satisfying the condition of Theorem 6 for $s=0$. Suppose G is not Hamiltonian. Using Menger's theorem for an arbitrary vertex $x_0 \in V(G)\setminus V(C)$ there exist $k \le \kappa$ vertex disjoint (except x_0) paths from x_0 to C. Let $v_1, v_2, \ldots, v_k$ denote the contact vertices of these paths on C and assume that they appear in this order according to a fixed arbitrary orientation on C. For a vertex $v \in V(C)$ let v^- and v^+ denote the vertex immediately preceding and succeeding v on C according to the orientation, respectively. Set $x_i = v_i^+$ for $1 \le i \le k$ and $y_i = v_{i+1}^-$ for $1 \le i \le k-1$ and $y_k = v_1^-$. It is well known (cf. [4]) that $X = \{x_0, x_1, \ldots, x_k\}$ is an independent set (see also Lemma 1(a)).

A closed segment on cycle C determined by $a, b \in V(C)$ is denoted by $[a,b]$ and a half-closed segment analogously by $[a,b)$ or $(a,b]$. If such a segment is traversed in the same orientation as C, then we write $[a,b]^+$, $[a,b)^+$, or $(a,b]^+$, and if it is traversed opposite to C, then we write $[a,b]^-$, $[a,b)^-$, or $(a,b]^-$ instead.

Lemma 1.

(a) $X = \{x_0, x_1, \ldots, x_k\}$ *is an independent set.*
(b) *If* $x_i, x_j \in X$, $1 \le i \le k$, *then there is no vertex* $z \in [x_i, y_i]^+$ *such that* $x_i z^+, x_j z \in E(G)$ *and* $x_j \in (x_i, y_i]^-$.
(c) *For* $x_i \in X$ *with* $1 \le i \le k$ *there is no vertex* $z \in V(C)$ *such that* $x_0 z, x_i z^+ \in E(G)$.

Proof.

(a) Adjacent vertices x_i and x_j or x_0 and x_i, $1 \le i < j \le k$, would imply the existence of a cycle $v_i x_0 [v_j, x_i]^- [x_j, v_i]^+$ or $v_i x_0 [x_i, v_i]^+$, respectively, being longer than C which contradicts the choice of C.
(b) The existence of such a vertex z would imply the existence of a cycle $v_i x_0 [v_j, z^+]^- [x_i, z]^+ [x_j, v_i]^+$ if $i = k$ or $1 \le i < j \le k$ or of a cycle $v_j x_0 [v_i, x_j]^- [z, x_i]^- [z^+, v_j]^+$ if $1 \le j < i \le k$, both cycles being longer than C.
(c) The existence of such a vertex z would imply the existence of a cycle $z x_0 [v_i, z^+]^- [x_i, z]^+$ or $z^+ [x_i, z]^+ x_0 [v_i, z^+]^-$ again being longer than C. □

The following sets are subsets of the vertex set of G:

$$A = V(G) \setminus N(X), \qquad A_C = A \cap V(C), \qquad A_R = A \setminus V(C),$$

that is, A is the set of vertices that are not adjacent to any vertex of X.

Lemma 2 (*See also [6]*)**.**

(a) $x_0 \in A_R$.
(b) *Two distinct elements of* X *have no common neighbor in* $V(G) \setminus V(C)$.

Proof.

(a) Since $x_0 \notin V(C)$ and X is independent, that is, $x_0 \notin N(X)$, we have $x_0 \in A_R$.
(b) If there were a common neighbor z of x_i and x_j or x_0 and x_i, $1 \le i < j \le k$, then the cycle $v_i x_0 [v_j, x_i]^- z [x_j, v_i]^+$ or $v_i x_0 z [x_i, v_i]^+$, respectively, would be longer than C. □

Let a_1 and a_2 be two consecutive vertices of A_C according to the orientation of C and $J = [a_1, a_2)$ the half-closed segment between a_1 and a_2 on C. Since $X \subset A$, we can suppose that a_1 and a_2 belong to a segment $[x_i, x_{i+1}]$ with $1 \le i \le k-1$ or to $[x_k, x_1]$. Let $N_J(x_i)$, $0 \le i \le k$, be the set of neighbors of x_i in J. Then, by Lemma 1(b) and (c), the neighborhoods

$$N_J(x_i), N_J(x_{i-1}), \ldots, N_J(x_1), N_J(x_k), N_J(x_{k-1}), \ldots, N_J(x_{i+1}), N_J(x_0)$$

or

$$N_J(x_k), N_J(x_{k-1}), \ldots, N_J(x_1), N_J(x_0),$$

respectively, form consecutive (possibly empty) segments of C in J that can only have their endvertices (extremities) in common. Hence, if a segment $J = [a_1, a_2)$ contains $t \ge 1$ vertices of $N(X)$ then

$$\sum_{i=0}^{k} d_J(x_i) \le k + t = \frac{k+1}{2}(t+1) - \frac{(k-1)(t-1)}{2} \le \frac{k+1}{2}(t+1). \tag{1}$$

Note that this upper bound also holds for $t = 0$. Now $V(C)$ can be partitioned into segments of type J. Using (1), Lemmas 1(a) and 2, we obtain the statement of the next lemma as follows.

If $|V(C)|$ is even, then $\sum_{i=0}^{k} d(x_i) \le |V(C)|\frac{k+1}{2} + (n-1-|V(C)|)$. If $|V(C)|$ is odd, then by parity arguments there is at least one segment with $t \ge 2$ vertices of $N(X)$. Using (1) implies $\sum_{i=0}^{k} d(x_i) \le |V(C)|\frac{k+1}{2} - \frac{k-1}{2} + (n-1-|V(C)|) = \left\lfloor \frac{|V(C)|}{2} \right\rfloor (k+1) + \left(n - 1 - 2\left\lfloor \frac{|V(C)|}{2} \right\rfloor\right)$.

Lemma 3. *Let C be a longest cycle in a k-connected non-Hamiltonian graph G of order n that satisfies the conditions of Theorem 6 for $s = 0$. Let $X = \{x_0, x_1 = v_1^+, \ldots, x_k = v_k^+\}$ where v_i, $1 \le i \le k$, are the contact vertices on C of k pairwise disjoint paths from $x_0 \notin V(C)$ to C. Then*

$$\sum_{i=0}^{k} d(x_i) \le \left\lfloor \frac{|V(C)|}{2} \right\rfloor (k+1) + \left(n - 1 - 2\left\lfloor \frac{|V(C)|}{2} \right\rfloor\right) = \left\lfloor \frac{|V(C)|}{2} \right\rfloor (k-1) + n - 1 \le (k+1)\frac{n-1}{2}.$$

Suppose G is not Hamiltonian. Then, by Theorem 4, $\mathrm{Cl}_n(G) = G'$ is not Hamiltonian. By Theorem 3, we obtain that $\alpha(G') \ge \kappa(G') + 1 \ge k + 1$.

Let $\{v_1, v_2, \ldots, v_{k+1}\}$ be an independent set of $k+1$ vertices of G'. We may assume that $d(v_1) \le d(v_2) \le \cdots \le d(v_{k+1})$. Then $d(v_k) \le \frac{n}{2} - 1$ since otherwise $d(v_k) + d(v_{k+1}) \ge n$ contradicting the definition of the n-closure. Hence

$$\sum_{i=1}^{k+1} d(v_i) = \sum_{i=1}^{k-1} d(v_i) + d(v_k) + d(v_{k+1}) \le (k-1)\frac{n-2}{2} + n - 1 = (k+1)\frac{n-2}{2} + 1.$$

By the condition of Theorem 6, we conclude that

$$\sum_{i=1}^{k+1} d(v_i) = (k+1)\frac{n-2}{2} + 1 \tag{2}$$

implying $d(v_i) = \frac{n-2}{2}$ for $1 \le i \le k$ and $d(v_{k+1}) = \frac{n}{2}$.

Suppose $\alpha(G') \ge k+2$. Let I be an independent set with $\alpha(G') = \alpha$ vertices $v_1, v_2, \ldots, v_\alpha$ such that $d(v_1) \le d(v_2) \le \cdots \le d(v_\alpha)$. By the same argument as before, we obtain $d(v_i) = \frac{n-2}{2}$ for $1 \le i \le k$ and $d(v_i) = \frac{n}{2}$ for $k+1 \le i \le \alpha(G')$. But then $d(v_i) + d(v_j) = n$ for $k+1 \le i < j \le \alpha(G')$, contradicting the definition of G'. Hence we have $\kappa(G') + 1 \le \alpha(G') = k+1$ which implies $k = \kappa(G')$.

We now consider the independent set X with $k+1$ vertices defined above. By (2) and Lemma 3, we have

$$(k+1)\frac{n-2}{2} + 1 = \sum_{i=0}^{k} d(x_i) \le \left\lfloor \frac{|V(C)|}{2} \right\rfloor (k+1) + \left(n - 1 - 2\left\lfloor \frac{|V(C)|}{2} \right\rfloor\right)$$

which implies $(k-1)\frac{n-2}{2} \le \left\lfloor \frac{|V(C)|}{2} \right\rfloor (k-1)$, that is, $\left\lfloor \frac{|V(C)|}{2} \right\rfloor \ge \frac{n-2}{2}$ and therefore $|V(C)| \ge n-2$.

1. $|V(C)| = n-1$:

We have $d(x_0) = \frac{n-2}{2}$ implying $k = \kappa(G') = \frac{n-2}{2}$ and $\alpha(G') = \frac{n}{2}$. But then, by the condition of the theorem and by Lemma 3, we have $\frac{1}{2}((k+1)(n-1)+3-k) = \frac{1}{2}(k+1)(n-2)+2 \le \sum_{i=0}^{k} d(x_i) \le \left\lfloor \frac{|V(C)|}{2} \right\rfloor (k+1) + \left(n - 1 - 2\left\lfloor \frac{|V(C)|}{2} \right\rfloor\right) = \frac{1}{2}(k+1)(n-2) + 1$, a contradiction.

2. $|V(C)| = n-2$:

Let C be a longest cycle of length $n-2$ and let x_0 and w be the two vertices not on C.

If x_0 and w are not adjacent, then $d(x_0) = d(w) = \frac{n-2}{2}$ and thus $N(x_0) = N(w)$. But then X is an independent set of $k+1$ vertices with $\sum_{i=0}^{k} d(x_i) \le (k+1)\frac{n-2}{2}$, a contradiction to the condition of the theorem.

Hence we may assume that $x_0 w \in E(G')$. Then $wx_i \notin E(G')$ for $1 \le i \le k$ by Lemma 2(b) which implies $d(x_i) + d(x_j) \le n-2$ for $1 \le i < j \le k$ (see [2]). Hence we have $d(x_i) = \frac{n-2}{2}$ for $1 \le i \le k$ and $d(x_0) = \frac{n}{2}$. Thus $k = \frac{n-2}{2}$ and so $\frac{1}{2}((k+1)(n-1)+3-k) = \frac{1}{2}(k+1)(n-2)+2 \le \sum_{i=0}^{k} d(x_i) \le \frac{1}{2}(k+1)(n-2)+1$, again a contradiction. □

2.b *Traceability* ($s = -1$)

For a graph G satisfying the condition of Theorem 6 for $s = -1$, we construct the graph $H \cong G + K_1$. Then H satisfies the hypothesis of Theorem 6 for $s = 0$. Hence H is Hamiltonian and therefore G is traceable. □

2.c *Hamiltonian-connectedness* ($s = 1$)

Let G be a graph that satisfies the condition of Theorem 6 for $s = 1$. Suppose there are two vertices u, w which are connected by a path P of maximum length that is not Hamiltonian. Using Menger's Theorem for an arbitrary vertex $x_0 \in V(G) \setminus V(P)$ there exist $k+1 \le \kappa$ vertex disjoint (except x_0) paths from x_0 to P. Let $v_1, v_2, \ldots, v_{k+1}$ denote the contact vertices of these paths on P and assume that they appear in this order according to a fixed orientation on P. Set $x_i = v_i^+$ for $1 \le i \le k$ and $x_{k+1} = v_{k+1}^+$ if $v_{k+1} \ne w$. Set $y_i = v_i^-$ for $2 \le i \le k+1$ and $y_1 = v_1^-$ if $u \ne v_1$.

The set $X = \{x_0, x_1, \ldots, x_k\}$ is independent (see proof of Lemma 4(a)). If x_{k+1} exists then by the same argument also $X \cup \{x_{k+1}\}$ is independent.

Lemma 4.

(a) *X is an independent set.*
(b) *If x_i and x_j are two distinct elements of $X \setminus \{x_0\}$ then there is no vertex $z \in [x_i, y_i]$ such that $x_i z^+, x_j z \in E(G)$.*
(c) *For $x_i \in X$ with $1 \le i \le k$ there is no vertex $z \in V(C)$ such that $x_0 z, x_i z^+ \in E(G)$.*

Proof.

(a) Adjacent vertices x_i and x_j or x_0 and x_i, $1 \leq i < j \leq k$, would imply the existence of a path $[u, v_i]^+ x_0 [v_j, x_i]^- [x_j, w]^+$ or $[u, v_i]^+ x_0 [x_i, w]^+$, respectively, being longer than P which contradicts the choice of P.
(b) The existence of such a vertex z would imply the existence of a path $[u, v_i]^+ x_0 [v_j, z^+]^- [x_i, z]^+ [x_j, w]^+$ if $1 \leq i < j \leq k$ or of a path $[u, v_j]^+ x_0 [v_i, x_j]^- [z, x_i]^- [z^+, w]^+$ if $1 \leq j < i \leq k$ both paths being longer than P.
(c) The existence of such a vertex z would imply the existence of a path $[u, z]^+ x_0 [v_i, z^+]^- [x_i, w]^+$ or $[u, v_i]^+ x_0 [z, x_i]^- [z^+, w]^+$ again being longer than P. □

Define $A = V(G) \setminus N(X)$, $A_P = A \cap V(P)$, and $A_R = A \setminus V(P)$.

Lemma 5.

(a) $x_0 \in A_R$.
(b) *Two distinct elements of X have no common neighbor in $V(G) \setminus V(P)$.*

Proof.

(a) Since X is independent, we have $X \subset A$ and therefore $x_0 \in A_R$.
(b) If there were a common neighbor z of x_i and x_j or x_0 and x_i, $1 \leq i < j \leq k$, then the path $[u, v_i]^+ x_0 [v_j, x_i]^- z [x_j, w]^+$ or $[u, v_i]^+ x_0 z [x_i, w]^+$, respectively, would be longer than P. □

Let a_1 and a_2 be two consecutive vertices of A_P according to the orientation of P and $J = [a_1, a_2)$ the half-closed segment between a_1 and a_2 on P. Since $X \subset A$, we can suppose that a_1 and a_2 belong to a segment $[x_i, x_{i+1}]$, $1 \leq i < k$, or to $[u, x_1]$ or to $[x_k, w]$. Then, by Lemma 4(b) and (c), the neighborhoods

$$N_J(x_i), N_J(x_{i-1}), \ldots, N_J(x_1), N_J(x_k), N_J(x_{k-1}), \ldots, N_J(x_{i+1}), N_J(x_0)$$

form consecutive (possibly empty) segments of P in J that can only have their endvertices (extremities) in common. Hence, if a segment $J = [a_1, a_2)$ contains $t \geq 1$ vertices of $N(X)$ then

$$\sum_{i=0}^{k} d_J(x_i) \leq (k+1) + (t-1) = k + t = \frac{k+1}{2}(t+1) - \frac{(k-1)(t-1)}{2} \leq \frac{k+1}{2}(t+1). \tag{3}$$

Note that this upper bound also holds for $t = 0$. For the convenience of the calculation, we add a vertex u_0 to $V(P)$ and the edge $u_0 u$. Then $u_0 \in A$. Now $V(P)$ can be partitioned into segments of type J. Using (3), Lemmas 4(a) and 5, we obtain the statement of the following lemma in the same way as Lemma 3 above. Note that we apply (3) to $V(P) \cup \{u_0\}$.

Lemma 6. *Let G be a $(k+1)$-connected graph of order n that satisfies the conditions of Theorem 6 for $s = 1$ but is not Hamiltonian-connected. If P is a longest path between vertices u and w of G that is not Hamiltonian, then*

$$\sum_{i=0}^{k} d(x_i) \leq \left\lfloor \frac{|V(P) \cup \{u_0\}|}{2} \right\rfloor (k+1) + \left(n + |\{u_0\}| - 1 - 2 \left\lfloor \frac{|V(P) \cup \{u_0\}|}{2} \right\rfloor \right) \leq (k+1)\frac{n-1}{2} + 1,$$

where $x_i = v_i^+$, $1 \leq i \leq k$, and v_i are the contact vertices on P of k pairwise disjoint paths from $x_0 \notin V(P)$ to P.

Since P is not Hamiltonian, $\mathrm{Cl}_{n+1}(G) = G'$ is not Hamiltonian-connected according to Theorem 4. By Theorem 3, we obtain that $\alpha(G') \geq \kappa(G') \geq k + 1$.

Let $I = \{v_1, v_2, \ldots, v_{\alpha(G')}\}$ be any independent set of $\alpha(G')$ vertices in G'. Then $d(v_i) + d(v_j) \leq n$ for $1 \leq i < j \leq \alpha(G')$ by definition of the closure. Assume that $d(v_1) \leq d(v_2) \leq \cdots \leq d(v_{\alpha(G')})$.

As in the Hamiltonicity case, we obtain for $\{v_1, v_2, \ldots, v_{k+1}\} \subseteq I$

$$\sum_{i=1}^{k+1} d(v_i) = (k+1)\frac{n-1}{2} + 1 \tag{4}$$

and therefore, $d(v_i) = \frac{n-1}{2}$ for $1 \leq i \leq k$ and $d(v_{k+1}) = \frac{n+1}{2}$ and $k < \frac{n-3}{2}$. If $\alpha(G') \geq k+1$ then $d(v_i) + d(v_j) = n + 1$ for $k + 1 \leq i < j \leq \alpha(G')$, a contradiction. Hence we have $k + 1 \leq \kappa(G') \leq \alpha(G') = k + 1$ which implies $k + 1 = \kappa(G')$.

We now consider the independent set X with $k + 1$ vertices defined above. By (4) and Lemma 6, we have

$$(k+1)\frac{n-1}{2} + 1 = \sum_{i=0}^{k} d(x_i) \leq \left\lfloor \frac{|V(P) \cup \{u_0\}|}{2} \right\rfloor (k+1) + n - 2 \left\lfloor \frac{|V(P) \cup \{u_0\}|}{2} \right\rfloor$$

which implies $(k-1)\frac{n-1}{2} \leq \left\lfloor \frac{|V(P) \cup \{u_0\}|}{2} \right\rfloor (k-1)$ and therefore $\left\lfloor \frac{|V(P) \cup \{u_0\}|}{2} \right\rfloor \geq \frac{n-1}{2}$ which gives $|V(P)| \geq n - 2$.

1. $|V(P)| = n - 1$:
We have $d(x_0) = \frac{n-1}{2}$ which implies $k + 1 = \kappa(G') = \alpha(G') = \frac{n-1}{2}$ and therefore $k = \frac{n-3}{2}$, a contradiction.

2. $|V(P)| = n-2$:

Let P be a longest path between vertices u and w of length $n-2$ and let x_0 and z be the two vertices not on P.

If x_0 and z are not adjacent, then $\frac{n-1}{2} \leq d(x_0) \leq \frac{n+1}{2}$. Repeating this consideration for z instead of x_0 and using Lemma 5(b), we obtain $d(x_0) = d(z) = \frac{n-1}{2}$ and $N_P(x_0) = N_P(z)$. But then $X \cup \{z\}$ is an independent set with $k+2 > \alpha(G')$ vertices, a contradiction.

If $x_0 z \in E(G)$ then $|N_P(x_0)| \geq \frac{n-1}{2} - 1 = \frac{n-3}{2}$. Suppose $|N_P(x_0)| = \frac{n-3}{2}$. Then, by combinatorial arguments, there is a segment J with $t = 3$ vertices of $N(X)$ or two segments J and $\bar{J}$ with $t = \bar{t} = 2$ vertices of $N(X)$. Then we obtain by using (3) and Lemma 6

$$\sum_{i=0}^{k} d(x_i) \leq (k+1)\frac{n-1}{2} + 1 - 2\frac{k-1}{2} < (k+1)\frac{n-1}{2} + \frac{1}{2},$$

a contradiction to the condition of the theorem. Hence we have $|N_P(x_0)| = \frac{n-1}{2}$ and analogously $|N_P(z)| = \frac{n-1}{2}$ and therefore $N_P(x_0) = N_P(z)$ by Lemma 5(b). This gives $k+1 = \kappa(G') = \alpha(G') = \frac{n-1}{2}$ and so $k = \frac{n-3}{2}$, a final contradiction. □

References

[1] J.-P. Bode, A. Kemnitz, I. Schiermeyer, A. Schwarz, Generalizing Bondy's theorems on sufficient conditions for Hamiltonian properties, Congressus Numer. 203 (2010) 5–13.
[2] J.A. Bondy, Longest paths and cycles in graphs of high degree, Research Report CORR 80-16, Department of Combinatorics and Optimization, Faculty of Mathematics, University of Waterloo, Waterloo, Ontario, Canada, 1980.
[3] J.A. Bondy, V. Chvátal, A method in graph theory, Discrete Math. 15 (1976) 111–135.
[4] V. Chvátal, P. Erdős, A note on Hamiltonian circuits, Discrete Math. 2 (1972) 111–113.
[5] G.A. Dirac, Some theorems on abstract graphs, Proc. Lond. Math. Soc. 2 (1952) 69–81.
[6] P. Fraisse, A new sufficient condition for Hamiltonian graphs, J. Graph Theory 10 (1986) 405–409.
[7] O. Ore, Note on Hamiltonian circuits, Amer. Math. Monthly 67 (1960) 55.
[8] O. Ore, Hamilton connected graphs, J. Math. Pures Appl. 42 (1963) 21–27.